Neutral Scalar Theory with Recoil*

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Using the fixed momentum-transfer dispersion relation without subtraction and introducing a cutoff in the dispersion integrals for "pion" energies larger than $(M\mu)^{1/2}$, the amplitude for scattering of "pions" (scalar and neutral, mass μ) from "nucleons" (spinless, mass $M\gg\mu$) has been calculated. The advantages gained by this approximation are primarily analytical: (a) The scattering amplitude can be solved in terms of s and p waves alone for pion energies $<(M\mu)^{1/2}$; (b) the full effect of the p waves and of their ambiguities on the s waves is displayed. The connection between the Castillejo-Dalitz-Dyson ambiguity and the unstable-particle interpretation has also been established in a special case.

I. INTRODUCTION

PARTIAL-WAVE dispersion relations for unequal mass particles are complicated by the presence of complex singularities and by the presence of a left-hand cut whose strength can only be determined after all the partial waves are known.[1–3] It would be interesting to find some model or approximation which leads to the scattering of a few partial waves and which might display the effect of the interconnection among the differential partial waves that is always present in a relativistic, crossing symmetric theory. Although such a model is bound to be deficient, it may still shed light on a problem whose solutions seems far in the future.

A guide to a suitable approximation is the observation that all static models lead to the scattering of a few partial waves.[4–7] Since a static model is physically equivalent to ignoring virtual processes of energy large compared to the target mass (when the latter is much larger than the mass of any other particle relevant to the analysis), the same simplification should result if cutoffs are introduced into the fixed momentum-transfer dispersion relation. In particular, for the scattering of a neutral scalar "pion" of mass μ from a spinless "nucleon" of mass M, the pole term of the fixed momentum transfer dispersion relation (without subtraction) yields comparable s- and p-wave amplitudes but d-wave scattering which is comparatively reduced at low energies by a factor of ω/M. Examination of this fixed momentum-transfer dispersion relation in the next section shows that when the dispersion integrals are dominated by the range $\mu\leq\omega<(M\mu)^{1/2}$, they admit solutions containing s and p waves alone in the same range. Only such solutions have been singled out in this work. In terms of Feynman diagrams, the range $\mu\leq\omega<(M\mu)^{1/2}$ corresponds to ignoring nucleon-antinucleon loops and to limiting the number of virtual pions. For such solutions the complex singularities of the partial-wave amplitudes are not present.

Using inelastic unitarity the s- and p-wave dispersion relations resulting from these approximations are shown to be generalized R functions.[8] The solutions of the integral equations for the partial-wave amplitudes contain a number of free parameters which include the unknown positions of the zeros of the amplitudes. If the latter are specified, the magnitude of the residues of the corresponding CDD poles cannot be arbitrarily large in this model. The CDD ambiguity of the p waves affects the upper bound of the residues of the s-wave CDD poles. By analytically continuing the partial-wave amplitude to the second Riemann sheet the correspondence between CDD poles and their unstable particle interpretation is also noted for this model.[9]

II. SCATTERING EQUATIONS

For the scattering of a neutral scalar pion from a spinless nucleon, the fixed momentum-transfer dispersion relation is

$$T(s,t)=g^2\left(\frac{1}{M^2-s}+\frac{1}{s+t-M^2-2\mu^2}\right)+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'$$
$$\times\operatorname{Im}T(s',t)\left(\frac{1}{s'-s-i\epsilon}+\frac{1}{s'+s+t-2M^2-2\mu^2}\right),\quad(1)$$

where s and t are the usual Mandelstam variables. If q and θ denote the center-of-mass three-momenta and angle for pion-nucleon scattering, then

$$s=((q^2+\mu^2)^{1/2}+(q^2+M^2)^{1/2})^2\equiv W^2\,,\qquad t=-2q^2(1-\cos\theta)\,.\quad(2)$$

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8 As defined in Ref. 3, a generalized R function of ζ is any function which is meromorphic in the cut ζ plane and whose imaginary part has the same sign as the imaginary part of ζ. Henceforth, we shall refer to a function with this property simply as an "R function."
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For our purposes it is more convenient to use the laboratory energy of the incident pion ω which is linearly related to the square of the center-of-mass energy s by the equation

$$s=M^2+\mu^2+2M\omega. \tag{3}$$

The connection between the center-of-mass momentum q and the laboratory energy ω is

$$q^2=\frac{\omega^2-\mu^2}{1+(\mu^2/M^2)+(2\omega/M)}. \tag{4}$$

The invariant amplitude is given in terms of the partial-wave amplitudes by the expansion

$$T(\omega,t)=\frac{W}{2q}\sum_L(2L+1)f_L(\omega)P_L\left(1+\frac{t}{2q^2}\right), \tag{5}$$

where

$$f_L(\omega)=\frac{e^{2i\,\mathrm{Re}\delta_L(\omega)}e^{-2\,\mathrm{Im}\delta_L(\omega)}-1}{2i}.$$

Evaluating $T(\omega,t)$ and its derivatives at $t=0$ from Eq. (1) and using Eq. (2) yields

$$T(\omega,0)=\frac{g^2\mu^2}{2M^2}\frac{1}{\omega^2-(\mu^2/2M)^2}+\frac{2}{\pi}\int_\mu^\infty\frac{d\omega'\,\omega'\,\mathrm{Im}T(\omega',0)}{\omega'^2-\omega^2-i\epsilon},$$

$$T'(\omega,0)=-\frac{g^2}{4M^2}\frac{1}{[\omega-\mu^2/2M]^2}+\frac{2}{\pi}\int_\mu^\infty\frac{d\omega'\,\omega'\,\mathrm{Im}T'(\omega',0)}{\omega'^2-\omega^2-i\epsilon}$$
$$-\frac{1}{2M}\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\,\mathrm{Im}T(\omega',0)}{(\omega'+\omega)^2} \tag{6}$$

$$T''(\omega,0)=\frac{g^2}{4M^3}\frac{1}{(\omega-\mu^2/2M)^3}+\frac{2}{\pi}\int_\mu^\infty\frac{d\omega'\,\omega'\,\mathrm{Im}T''(\omega',0)}{\omega'^2-\omega^2-i\epsilon}$$
$$-\frac{1}{M}\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\,\mathrm{Im}T'(\omega',0)}{(\omega'+\omega)^2}$$
$$+\frac{1}{2M^2}\frac{1}{\pi}\int_\mu^\infty\frac{d\omega'\,\mathrm{Im}T(\omega',0)}{(\omega'+\omega)^3}.$$

The basic assumption of this work is that when M becomes very large, those solutions of Eq. (6) which satisfy

$$\omega MT''(\omega,0)\sim T'(\omega,0)\sim T(\omega,0)/\mu^2 \tag{7}$$

are the solutions of physical interest. This assumption is based on a comparison of the pole terms of the above equations. The solutions for $T(\omega,0)$ and $T'(\omega,0)$ obtained on this basis are readily shown to contribute terms proportional to $1/M^5$ when substituted into the last two integrals on the right-hand side of the equation for $T''(\omega,0)$. We cannot exclude solutions violating Eq. (7) on mathematical grounds alone; however, for energies $\omega<(M\mu)^{1/2}$ this choice results in d and higher partial waves which are at least $(\mu/M)^{1/2}$ times smaller than the s or p waves, as will be made clear below. This is a reasonable expectation for a cutoff model with large nucleon mass since all known static theories lead to the scattering of a few partial waves. Since

$$T(\omega,t)=[T(\omega,0)+tT'(\omega,0)]+[T(\omega,t)-T(\omega,0)-tT'(\omega,t)],$$

a comparison of the first and last terms in the range of convergence of the power series in t reveals from Eq. (7) that

$$|[T(\omega,t)-T(\omega,0)-tT'(\omega,0)]|\lesssim|tT'(\omega,0)|\times\left(\frac{\omega^2-\mu^2}{M\omega}\right),$$

or, equivalently,

$$T(\omega,t)=T(\omega,0)+tT'(\omega,0)+O\left[tT'(\omega,0)\left(\frac{\omega^2-\mu^2}{M\omega}\right)\right]. \tag{8}$$

The contributions of $T'''(\omega,0)$ and higher derivatives to the right-hand side of Eq. (8) involve higher powers of $(\omega^2-\mu^2)/M\omega$ which is a small quantity for the energy range of interest in the following treatment. Since closed nucleon loops have been ignored, the unsubtracted dispersion relations assure that the power series in t converges up to $4\,M\mu$[10]; however, since $|t|<4q^2<4(\omega^2-\mu^2)$ in the physical region, it follows that for $\omega^2<M\mu$ the power series in t will converge for all scattering angles. By cutting off the dispersion integrals above $(M\mu)^{1/2}$ [keeping less than $(M\mu)^{1/2}$ pions in intermediate states], it will suffice to keep the first two terms of Eq. (8). For $\omega^2<M\mu$, keeping s, p, and d waves in Eq. (5) and comparing with Eq. (8) reveals that the d and higher partial-wave amplitudes are $(\mu/M)^{1/2}$ times smaller than the s- and p-wave amplitudes. For this reason we confine our attention to the energy range $\omega<(M\mu)^{1/2}$ and retain only s and p partial-wave amplitudes.

From Eqs. (6) and for $M\gg\mu$, it follows that

$$T(\omega,0)=\frac{g^2\mu^2}{2M^2\omega^2}+\frac{1}{\pi}\int_{\mu^2}^\infty\frac{d\omega'^2\,\mathrm{Im}T(\omega',0)}{\omega'^2-\omega^2-i\epsilon}, \tag{9}$$

$$T'(\omega,0)=-\frac{g^2}{4M^2\omega^2}+\frac{1}{\pi}\int_{\mu^2}^\infty\frac{d\omega'^2\,\mathrm{Im}T'(\omega',0)}{\omega'^2-\omega^2-i\epsilon}, \tag{10}$$

which is the basis of the development to follow. From Eqs. (4), (5), and (8) we see that for $\omega^2<M\mu$

$$T(\omega,0)=(W/2q)[f_0(\omega)+3f_1(\omega)], \tag{11}$$

$$T'(\omega,0)=(3W/4q^3)f_1(\omega). \tag{12}$$

The usefulness of fixed momentum-transfer dispersion relations in obtaining partial-wave dispersion relations

[10] From the Mandelstam representation the absorptive part of the amplitude $\mathrm{Im}T(\omega,t)$ is real provided that $-4\mu M<t<4\mu^2$. When $N\bar{N}$ loops are excluded, this range is increased to $-4\mu M<t<4M^2$. A convergent power series expansion in t is restricted by the corresponding intervals.

must be attributed to the dominance of low-energy contributions to the integrals of Eqs. (9) and (10), because a partial-wave decomposition of $T(\omega,t)$ is practical only so long as a manageable expression for $T(\omega,t)$ for $0<-t<4q^2$ can be obtained. For instance, when $\omega>M$ the first two terms of Eq. (8) may not be sufficient, which also then invalidates Eqs. (11) and (12). After the introduction of inelastic unitarity in the following section we shall see that it can be also used as a cutoff in the spirit of our approximation.

III. SOLUTION FOR P-WAVE AMPLITUDE

The right-hand side of Eq. (10) defines an analytic function

$$T_1(\zeta)=-\frac{g^2}{4M^2\zeta}+\frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{d\omega'^2\,\mathrm{Im}T'(\omega',0)}{\omega'^2-\zeta},\qquad(13)$$

such that for $\omega^2>\mu^2$

$$\lim_{\epsilon\to 0+}T_1(\omega^2+i\epsilon)=T'(\omega,0).\qquad(14)$$

Since $\mathrm{Im}f_L\geq 0$, it follows from (5) that $\mathrm{Im}T'(\omega,0)\geq 0$. Hence, it is readily seen that $T_1(\zeta)$ is an R function of ζ. Defining for $\omega\geq\mu$,

$$\frac{\mathrm{Im}T'(\omega,0)}{|T'(\omega,0)|^2}\equiv\frac{4q^3}{3W}R_1(\omega),\qquad(15)$$

it follows from Eqs. (5) and (12) that, for $\omega^2<M\mu$, $R_1(\omega)$ is the ratio of total to elastic p-wave cross section. For $\omega^2\geq M\mu$ a simple interpretation of $R_1(\omega)$ cannot be given since $T'(\omega,0)$ is no longer the p-wave amplitude. However, in this region $R_1(\omega)$ affords a natural way of introducing the necessary cutoff on the integrals.

$T_1(\zeta)$ has the following properties:

(i) $T_1(\zeta)$ is a real analytic function, $T_1(\zeta^*)=T_1^*(\zeta)$, except for a cut for real ζ in the interval (μ^2,∞).

(ii) As $\zeta\to\infty$ away from the real axis, $T_1(\zeta)$ tends to zero as $1/\zeta$.

(iii) $T_1(\zeta)$ is an R function of ζ. Thus, $T_1(\zeta)$ can have zeros only on the real axis. We denote the position of a zero by ζ_i^1, $i=1,\cdots,n$. Inspection of Eq. (13) reveals that $\zeta_i^1>0$. Since the derivative of the integral term of Eq. (13) is positive in the interval $(0,\mu^2)$, there can be at most one ζ_i^1 in this interval. Furthermore, $T_1(\zeta)$ can develop no poles in this interval.

(iv) $T_1(\zeta)$ has a simple pole at $\zeta=0$ with residue $(-g^2/4M^2)$.

We briefly construct the solution for $T_1(\zeta)$ using the method previously given by CDD. Introducing the function

$$H_1(\zeta)\equiv -1/T_1(\zeta),\qquad(16)$$

which is also an R function, it follows from Eq. (15) for real $\zeta\neq\zeta_i^1$ that

$$\mathrm{Im}H_1(\omega^2)=\rho_1(\omega)R_1(\omega),\qquad(17)$$

where $\rho_1(\omega)\equiv 4q^3/3W$. Referring to the CDD paper, the most general solution for $H_1(\zeta)$ is

$$H_1(\zeta)=A_1\zeta+B_1+\frac{C_1}{\zeta}+\frac{\zeta}{\pi}\int_{\mu^2}^{\infty}\frac{d\omega'^2\rho_1(\omega')R_1(\omega')}{\omega'^2(\omega'^2-\zeta)}+\zeta\sum_{i=1}^{n}\frac{\mathcal{R}_i^1}{\zeta_i^1(\zeta_i^1-\zeta)},\qquad(18)$$

where A_1, B_1, C_1 are real and $A_1\geq 0$, $C_1\leq 0$, $\mathcal{R}_i^1>0$. To satisfy property (iv), we must demand that $H_1(\zeta)|_{\zeta=0}=0$ and $dH_1(\zeta)/d\zeta|_{\zeta=0}=4M^2/g^2$. Imposing these conditions leads to the following values for constants of Eq. (18):

$$B_1=0,\quad C_1=0,$$

$$A_1=\frac{4M^2}{g^2}-\int_{\mu^2}^{\infty}\frac{d\omega'^2\rho_1(\omega')R_1(\omega')}{\omega'^4}-\sum_i\frac{\mathcal{R}_i^1}{(\zeta_i^1)^2}.\qquad(19)$$

Since, in order that Eq. (18) be an admissible solution to the integral equation (13), A_1 cannot be negative, there exists a weak interrelationship between physically allowed values of g^2, $R_1(\omega)$, and the $\mathcal{R}_i^1$ through Eq. (19).

For ζ in the interval $(0,\mu^2)$ the derivative of $H_1(\zeta)$ is strictly positive. If one ζ_i^1 does lie in this interval, then it is easily seen from property (iii) that its residue $\mathcal{R}_i^1$ must be such that $H_1(\mu^2)\leq 0$.

It is noted from Eq. (13) that

$$T_1(\zeta)+g^2/4M^2\zeta\qquad(20)$$

is also an R function. A solution $T_1(\zeta)$ which itself is an R function does not necessarily satisfy this requirement. This may place additional restrictions on the residues $\mathcal{R}_i^1$. For example, the condition that (20) be an R function in the neighborhood of CDD poles results in the requirement

$$4M^2/g^2\geq\mathcal{R}_i^1/(\zeta_i^1)^2,$$

which, however, is already contained in Eq. (19) since $A_1\geq 0$.

In the vicinity of a CDD pole the derivative of the amplitude is positive definite since

$$dT_1(\zeta)/d\zeta|_{\zeta=\zeta_i^1}=1/\mathcal{R}_i^1.\qquad(21)$$

Furthermore, for $\mu^2\leq\zeta<M\mu$, the real and imaginary parts of T_1 are related to the phase shifts by

$$\mathrm{Re}T_1(\zeta)=\frac{3W}{8q^3}\sin 2\,\mathrm{Re}\delta(\zeta)e^{-2\,\mathrm{Im}\delta(\zeta)},$$
$$\mathrm{Im}T_1(\zeta)=\frac{3W}{8q^3}[1-\cos 2\,\mathrm{Re}\delta(\zeta)e^{-2\,\mathrm{Im}\delta(\zeta)}].\qquad(22)$$

Thus, $\mathrm{Re}T_1(\zeta)$ must pass from negative to positive values whenever $T_1(\zeta)=0$ and the real part of the phase shift passes through a multiple of π. The inelastic phase shifts also vanish at the point $\zeta=\zeta_i^1$. In going from one singularity ζ_i^1 to the next, the real part of the phase shift increases by a multiple of π (not necessarily π as is the case in models which allow no inelastic process).

IV. SOLUTION FOR S-WAVE AMPLITUDE

We introduce the function

$$T_0(\omega)=T(\omega,0)-2q^2T'(\omega,0), \tag{23}$$

which for $\mu^2\leq\omega^2<M\mu$ is trivially related through Eqs. (11) and (12) to the s-wave amplitude,

$$T_0(\omega)=(W/2q)f_0(\omega). \tag{24}$$

Substituting Eq. (23) into Eq. (9) and making use of Eq. (10), the integral equation (9) can be cast into the following form:

$$T_0(\omega)=\frac{1}{1+2\omega/M}\left[\frac{g^2}{2M^2}\left(1+\frac{2\mu^2}{\omega M}\right)+\frac{2}{\pi}\int_{\mu^2}^{\infty}\frac{d\omega'^2\,\mathrm{Im}T'(\omega',0)}{(1+2\omega'/M)}\left(1+\frac{2}{M}\frac{\omega\omega'+\mu^2}{\omega'+\omega}\right)\right]+\frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{d\omega'^2\,\mathrm{Im}T_0(\omega')}{\omega'^2-\omega^2-i\epsilon}. \tag{25}$$

Letting

$$\mathrm{Im}T_0(\omega)/|T_0(\omega)|^2=\rho_0(\omega)R_0(\omega)\text{ for }\omega\geq\mu, \tag{26}$$

where $\rho_0(\omega)\equiv2q/W$, it follows that $R_0(\omega)$ can be interpreted as the ratio of the total to the elastic s-wave cross section for $\mu^2\leq\omega^2<M\mu$. With the assumption that the region $\omega^2<M\mu$ dominates the dispersion integrals, a good approximation to the inhomogeneous terms of the integral equation (25) in this range is the constant

$$D=\frac{g^2}{2M^2}+\frac{2}{\pi}\int_{\mu^2}^{\infty}d\omega'^2\,\mathrm{Im}T'(\omega',0). \tag{27}$$

This approximation introduces errors for $\omega\gtrsim M$ which are not of concern since outside the range $\mu<\omega<(M\mu)^{1/2}$, $T_0(\omega)$ is not simply related to the s-wave amplitude. Thus, one can define a function

$$T_0(\zeta)=D+\frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{d\omega'^2\rho_0(\omega')R_0(\omega')|T_0(\omega')|^2}{\omega'^2-\zeta}, \tag{28}$$

such that for $M\mu>\omega^2\geq\mu^2$, $\lim_{\epsilon\to0+}T_0(\omega^2+i\epsilon)$ is the function of physical interest. The solution to Eq. (27) is obtained by the introduction of the function

$$H_0(\zeta)\equiv-1/T_0(\zeta), \tag{29}$$

where again both $H_0(\zeta)$ and $T_0(\zeta)$ are R functions. Since the high-energy part of the dispersion integrals is assumed to be small,

$$\lim_{\zeta\to\infty}T_0(\zeta)=D.$$

If the scattering amplitude has zeros at ζ_i^0 (note that all $\zeta_i^0>\mu^2$ since $D>0$), then the solution having the required analyticity and obeying unitarity is

$$H_0(\zeta)=B_0+\frac{\zeta}{\pi}\int_{\mu^2}^{\infty}\frac{d\omega'^2\rho_0(\omega')R_0(\omega')}{\omega'^2(\omega'^2-\zeta)}+\zeta\sum_i\frac{\mathcal{R}_i^0}{\zeta_i^0(\zeta_i^0-\zeta)}, \tag{30}$$

where

$$B_0=\frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{d\omega'^2\rho_0(\omega')R_0(\omega')}{\omega'^2}+\sum_i\frac{\mathcal{R}_i^0}{\zeta_i^0}-\frac{1}{D}. \tag{31}$$

Since $T_0(\zeta)>0$ and $dT_0(\zeta)/d\zeta>0$ for real ζ in the interval $(0,\mu^2)$, it suffices to choose

$$B_0+\frac{\mu^2}{\pi}\int_{\mu^2}^{\infty}\frac{d\omega'^2\rho_0(\omega')R_0(\omega')}{\omega'^2(\omega'^2-\mu^2)}+\mu^2\sum_i\frac{\mathcal{R}_i^0}{\zeta_i^0(\zeta_i^0-\mu^2)}<0. \tag{32}$$

Combining Eqs. (32), (31), (17), and (16), the explicit form of this restriction is

$$\left[\frac{g^2}{2M^2}+\frac{2}{\pi}\int_{\mu^2}^{\infty}\frac{d\omega'^2\rho_1(\omega')R_1(\omega')}{|H_1(\omega'^2+i\epsilon)|^2}\right]^{-1}>\frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{d\omega'^2\rho_0(\omega')R_0(\omega')}{\omega'^2-\mu^2}+\sum_i\frac{\mathcal{R}_i^0}{\zeta_i^0-\mu^2}. \tag{33}$$

The CDD ambiguity of the s waves $(\mathcal{R}_i^0,\zeta_i^0)$ is influenced through Eqs. (18) and (33) by the CDD ambiguity of the p waves $(\mathcal{R}_i^1,\zeta_i^1)$. If the positions of the respective poles ζ_i^1, ζ_i^0 are regarded as fixed parameters, then larger values of the $\mathcal{R}_i^1$ allow larger values of the $\mathcal{R}_i^0$.

From Eq. (28) it is easily seen that

$$\zeta(T_0(\zeta)-D) \tag{34}$$

is also an R function. The consequence of this requirement is easily established for ζ in the neighborhood of ζ_i^0, for example, and yields

$$1/D>\mathcal{R}_i^0/\zeta_i^0, \tag{35}$$

which is already contained in the inequality (32). Whether or not additional restrictions follow from the requirement that (34) be an R function has not been established.

The solutions for T_1 and T_0 given by Eqs. (18) and (30) do not tend uniformly in the limit of infinite nucleon mass to the CDD solution of the recoilless model. However, the CDD s-wave solution can be obtained from the integral equations (13) and (28) by taking the infinite M limit and choosing the particular p-wave solution $T_1=0$.

V. CONTINUATION OF THE SCATTERING AMPLITUDE TO THE SECOND RIEMANN SHEET

According to Peierl's suggestion, unstable particles may be associated with poles of the scattering amplitude in the lower half plane of the second Riemann sheet.[11] We investigate this association within the framework of the p-wave solution of our model. Since $H_1(\zeta)$ is an R function, it cannot have zeros on the first sheet off the real axis.

The second sheet is defined by

$$H_1^{\rm II}(\zeta+i\epsilon)\equiv H_1(\zeta-i\epsilon)=H_1(\zeta+i\epsilon)-2i\rho_1(\zeta) \quad (36)$$

for real ζ in the interval (μ^2,E_I^2) where E_I is the inelastic threshold.[12] It is consistent with our previous arguments to approximate $\rho_1(\zeta)\equiv 4q^3/3W$ in this interval by $4(\omega^2-\mu^2)^{3/2}/3M$. It is possible to construct $H_1^{\rm II}(\zeta)$ in the whole complex ζ plane through the equation

$$H_1^{\rm II}(\zeta)=H_1(\zeta)-i\frac{8(\zeta-\mu^2)^{3/2}}{3M}. \quad (37)$$

As pointed out by Chew,[13] if $\mathcal{R}_i^1$ is not too large, then the rapid variation of $H_1(\zeta)$ in the neighborhood of a CDD pole should produce a zero of $H_1^{\rm II}(\zeta)$ near ζ_i^1. As is clear from Eq. (19), very large values of the $\mathcal{R}_i^1$ are not compatible unless associated with large ζ_i^1. The location of the zeros of $H_1^{\rm II}(\zeta)$ for intermediate values of the $\mathcal{R}_i^1$ is a detailed question which will also involve a knowledge of the ratio of the total to elastic p-wave cross section.

Let the zeros of $H_1^{\rm II}(\zeta)$ be at $\zeta=w_i^2$. We define in analogy with Eq. (13) and (16) the coupling constant g_i of an unstable particle through

$$\left.\frac{dH_1^{\rm II}(\zeta)}{d\zeta}\right|_{\zeta=w_i^2}=\frac{4M^2}{g_i^2}. \quad (38)$$

It should be noted that, in general, w_i^2 and g_i^2 as defined in this manner are complex numbers. This definition of the coupling constant squared as the residue of the pole on the second Riemann sheet differs from that given in Ref. 9. However, for poles near the real axis, ${\rm Re}g_i^2$ corresponds to their definition of the coupling constant. For the case that we analyze below, only the real part of g_i^2 turns out to be of physical interest. From Eqs. (18), (37), and (38)

$$\frac{4M^2}{g_i^2}=A_1+\frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{d\omega'^2\rho_1(\omega')R_1(\omega')}{(\omega'^2-w_i^2)^2}+\sum_j\frac{\mathcal{R}_i^1}{(\zeta_j^1-w_i^2)^2}-4i\frac{(w_i^2-\mu^2)^{1/2}}{M}, \quad (39)$$

which by partial integration on ω'^2 becomes

$$\frac{4M^2}{g_i^2}=A_1+\frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{d\omega'^2}{\omega'^2-w_i^2}\cdot\frac{d^2}{d\omega'^2}(\zeta_1(\omega')R_1(\omega'))+\sum_j\frac{\mathcal{R}_j^1}{(\zeta_j^1-w_i^2)^2}-4i\frac{(w_i^2-\mu^2)^{1/2}}{M}, \quad (40)$$

since ρ_1 and R_1 vanish at the lower and upper limits of integration, respectively. Inspection of Eq. (40) reveals that the real part of g_i^2 is not necessarily positive.

If the position of the zero w_i^2 of $H_1^{\rm II}$ is sufficiently close to a pole ζ_i^1 of $H_1^{\rm I}$ then the CDD pole term will dominate the right-hand side of Eq. (40) and ${\rm Re}g_i^2$ will be positive.

To investigate the identification of a CDD pole with the presence of an unstable particle in the case of narrow partial width, we write $w_i^2=x+iy$, where by definition

$$H_1^{\rm II}(w_i^2)=0. \quad (41)$$

Assuming $y\ll x$, Eq. (41) yields

$$H_1^{\rm II}(x+i\epsilon\theta(y))+iy\frac{d}{dx}H_1^{\rm II}(x+i\epsilon\theta(y))=0, \quad (42)$$

where $\epsilon\to 0+$. From Eq. (42) it follows that

$${\rm Im}H_1^{\rm II}(x+i\epsilon\theta(y))=-y\,{\rm Re}\frac{d}{dx}H_1^{\rm II}(x+i\epsilon\theta(y)), \quad (43)$$

and from Eqs. (43), (36), and (18) we obtain

$${\rm Im}H_1^{\rm II}(x+i\epsilon\theta(y))=-\rho_1(x)[2-R_1(x)]\theta(y), \quad (44)$$

which in view of earlier remarks can be reduced to

$${\rm Im}H_1^{\rm II}(x+i\epsilon\theta(y))=-4(x-\mu^2)^{3/2}/3M\times[2-R_1(x)]\theta(y). \quad (45)$$

From Eq. (38) we have the approximate relation

$${\rm Re}(d/dx)H_1^{\rm II}(x+i\epsilon\theta(y))={\rm Re}4M^2/g_i^2 \quad (46)$$

and combining this with Eqs. (43) and (45) gives

$$y=({\rm Re}g_i^2/3M^3)(x-\mu^2)^{3/2}[2-R_1(x)]\theta(y). \quad (47)$$

[11] R. E. Peierls, in *Proceedings of the 1954 Glasgow Conference on Nuclear and Meson Physics* (Pergamon Press Inc., New York, 1955), p. 296.

[12] If the continuation to the second sheet had been made across the branch cut above the inelastic threshold, then the factor $R_1(\zeta)$ would appear in the last term of Eq. (36). The analytic properties of $R_1(\zeta)$ are unknown and might very well preclude the construction of such a $H_1^{\rm II}(\zeta)$ in the whole complex ζ plane. In fact, we know that $R_1(\zeta)$ cannot be analytic in the entire ζ plane since it is constant in the interval (μ^2,E_I^2).

[13] G. F. Chew, Lawrence Radiation Laboratory Report, UCRL-9289, 1960 (unpublished), p. 56. See also Sec. 11 of Ref. 2.

$R_1=1$ at a CDD pole; if the residue of the latter is small, x can be expected to be near the position of the pole and $R_1(x)\approx 1$. Consequently, for the consistency of Eq. (47) we see that a zero of H_1^{II} near the real axis can exist only if $\mathrm{Re}g_i^2$ is positive. Furthermore, if H_1^{II} vanishes at $x+iy$ for $y\ll x$, then from Eq. (47) it also has a zero at the complex conjugate point $x-iy$.[14] The real part of the Eq. (42) gives an implicit equation for the value of x. The square of the "mass" corresponding to this pole is

$$m_i^2=M^2+\mu^2+2M\left(x^{1/2}+\frac{iy}{2x^{1/2}}\right). \tag{48}$$

Since the time development of any quantum mechanical system is given by e^{-imit}, the identification of $y/x^{1/2}$ with the inverse lifetime of an unstable particle follows for the zero of H_1^{II} with $y<0$.

The foregoing considerations apply to CDD poles which cause the scattering amplitude to develop zeros on the second sheet at points ζ for which $\mathrm{Re}\zeta>\mu^2$. Since $H_1^{\mathrm{II}}(x+i\epsilon)$ is real for $0<x<\mu^2$, we conclude from Eq. (43) that $y=0$ at a zero of H_1^{II} with $0<\mathrm{Re}\zeta<\mu^2$.

Although the presence of a CDD pole term in the inverse amplitude H_1 may cause H_1^{II} to develop a zero, the correspondence may not be one to one. In other words, H_1^{II} may have extra zeros which are not directly associated with the CDD ambiguity. For instance, in the Lee-Serber solution for the charged scalar theory (no CDD ambiguity present) the scattering amplitude has a pole on the second sheet at a point on the real axis in the interval $(0,\mu^2)$. Such zeros arise from the nature of the forces and are not of kinematical origin.[15]

VI. DISCUSSION AND CONCLUSIONS

The approximation that is central to all the foregoing is the neglect of processes of energy $\omega\geq(M\mu)^{1/2}$; hence, our results are only valid for $\mu\leq\omega<(M\mu)^{1/2}$. Since we are considering large values of M, this includes a considerable range of energies. The description of the s- and p-wave amplitudes in terms of R functions is clearly the result of our approximations since, in general, the partial-wave amplitudes are known to have complex singularities on the first Riemann sheet. Nevertheless, the strength of the complex singularities may not be sufficient to change drastically our results for $\mu\leq\omega<(M\mu)^{1/2}$.

For given zeros of the p-wave amplitude the residues of the corresponding CDD poles cannot be arbitrarily large as seen from Eq. (19). Although CDD poles with small residues (narrow widths) may be associated with the presence of unstable particles, it is not clear whether they lend themselves to a simple interpretation when the residues become as large as allowed. At a CDD pole the real part of the phase shift passes through a multiple of π and the imaginary part of the phase shift vanishes. Thus, with an unstable particle interpretation of the CDD ambiguity, the partial-wave amplitude in the corresponding angular momentum channel must vanish near the energy at which the unstable particle is produced.

The authors thank Dr. P. Signell for enlightening comments about singularities on the second Riemann sheet.

[14] R. Jacob and R. G. Sachs, Phys. Rev. **121**, 350 (1961). The singularities on the second Riemann sheet of the propagator considered in this paper also occur symmetrically placed with respect to the real axis.

[15] A. O. Barut and K. H. Ruei, Phys. Rev. **122**, 1340 (1961).